

# Managing Interference Correlation Through Random Medium Access

Yi Zhong, Wenyi Zhang, *Senior Member, IEEE* and Martin Haenggi, *Senior Member, IEEE*

**Abstract**—The capacity of wireless networks is fundamentally limited by interference. However, little research has focused on the interference correlation, which may greatly increase the local delay (namely the number of time slots required for a node to successfully transmit a packet). This paper focuses on the question that whether increasing randomness in the MAC, such as frequency-hopping multiple access (FHMA) and ALOHA, helps to reduce the effect of interference correlation. We derive closed-form results for the mean and variance of the local delay for the two MAC protocols and evaluate the optimal parameters that minimize the mean local delay. Based on the optimal parameters, we propose the definitions of two operation regimes: correlation-limited regime and bandwidth-limited regime. Our results reveal that while the mean local delays for FHMA with  $N$  sub-bands and for ALOHA with transmit probability  $p$  are the same when  $p = \frac{1}{N}$  with thermal noise ignored, significant difference exists between the variances. At last, we evaluate the mean delay-jitter tradeoff and the bounds on the tail probability of the local delay, which shed key insights into the system design.

**Index Terms**—ALOHA, frequency-hopping, interference correlation, local delay, Poisson point process, stochastic geometry.

## I. INTRODUCTION

### A. Motivation

A main limitation to the capacity of wireless communication systems is interference, which is a function of the locations of the transmitters, path loss and fading. The interference has been studied extensively in the literature; however, much less

attention has been paid to the topic of interference correlation until recently. It is well recognized that correlated fading reduces the performance gain in multi-antenna communications [1]. Likewise, it is also proved that interference correlation decreases the diversity gain [2]. Intuitively, interference correlation may come from correlated channel attenuation, like correlated fading and shadowing, but more importantly, correlation stems from the spatial distribution of transmitters and MAC protocols. The lines of recent research can be divided into three categories based on different configurations for the receiver:

- **Correlation between different time slots:** Assume that the receiver is equipped with single antenna. This line of research explored the interference correlation at the same receiver between different time slots. Related works include [3]–[5].
- **Correlation between different receive antennas:** In this case, it is assumed that the receiver is equipped with co-located multiple antennas and the antennas experience the same large scale path loss but independent fading. The correlation between different antennas exists because the interferences received by different antennas come from the same source of transmitters. Related works include [2].
- **Correlation between different receivers:** This refers to the interference correlation between different receivers which are many wavelengths apart. Since the network may make use of relay and cooperative transmission, it is necessary to take this type of interference correlation into consideration for an accurate analysis. Related works include [5].

In this work, we focus on the interference correlation between different time slots at the same receiver. The interferences of two different time

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slots are correlated because they come from the same set of transmitters and the fading, shadowing and traffic may also be correlated. To facilitate the analysis, we only focus on the correlation caused by the spatial distribution of transmitters, and assume that fading, shadowing and traffic are independent. This type of correlation brings about the fact that if the transmission fails at the previous time slot, there is a significant probability that the transmission will also fail in the next few time slots [5]. Thus a simple retransmission mechanism may not be an effective method. The most direct impact of this type of correlation is the increase of the local delay. Local delay is defined as the number of time slots required by a node to successfully transmit a packet to its next-hop node<sup>1</sup>. With temporal correlation between different time slots, the mean local delay will inevitably be larger than in the completely independent case.

In view of this, we consider some form of man-made randomization by introducing MAC dynamics to reduce the correlation. The following analysis will be carried out in parallel under two different kinds of MAC protocols:

- **FHMA (Frequency-hopping multiple access):** FHMA is simply implemented by dividing the entire frequency band into a  $N$  sub-bands and letting each transmitter choose a sub-band uniformly randomly in each time slot. We focus on slow frequency-hopping, i.e., hopping at the time scale of a time slot, not at the time scale of a symbol. There are three benefits by splitting the entire frequency bands into sub-bands. First and foremost, it increases the uncertainty, thereby reducing the effect of interference correlation between different time slots. Second, the interference for a given transmission is significantly reduced because the intensity of the interfering transmitters are scaled by  $\frac{1}{N}$ . Third, the noise power is also reduced by  $N$  since each transmission occurs in a more narrow band. However, on the other side, splitting into sub-bands scales down the rate at the same time.
- **ALOHA:** In ALOHA, if a packet is to be transmitted during a time slot, the packet will

only be transmitted with a given probability using the entire frequency band. Decreasing the transmit probability increases the uncertainty and reduces the interference, while the noise power will not be reduced. Nevertheless, if the transmit probability is small, it is less likely that a packet will be transmitted in a time slot, which may increase the local delay.

Since FHMA is often viewed as a spread-spectrum scheme, it is valuable to mention DS-CDMA. For synchronous orthogonal CDMA like those using Walsh codes, a receiver can in theory completely reject arbitrarily strong signals from the transmitters using different spreading sequences; thus, only those transmitters using the same spreading sequence as the desired link will cause interference, similar to the FHMA case where only those transmitters using the same sub-band as the desired link will cause interference. If the spreading sequence is randomly chosen for each transmission, the analysis and results of the local delay are exactly the same as that in FHMA. For asynchronous CDMA using pseudo-noise (PN) sequences, the interference comes from all transmitters and is usually approximated as Gaussian noise in the literature. The works in [7] and [8] have discussed the difference between asynchronous CDMA and FHMA in terms of outage probability and throughput. In asynchronous CDMA, although the desired signal is increased by the processing gain, the interference still comes from all transmitters. Therefore, the analysis of the local delay for asynchronous CDMA is the same as that for FHMA with  $N=1$ , i.e., when no bandwidth splitting is employed. We will show that in this case, the distribution of the local delay has a heavy tail, which results in an infinite mean local delay.

## B. Contributions

In this work, we focus on the question that whether increasing randomness in the MAC helps to decrease the local delay. We apply the so-called Poisson bipolar model, see [9, Sec. 5.3], and derive the means and variances of the local delay under two different MAC protocols: FHMA and ALOHA. To the best of our knowledge, the variance of local delay, which is an important measure that characterizes the fluctuation of local delay, has not been explored in the existing work. Based on the means

<sup>1</sup>The definition of local delay in our work is consistent with [6] in which the local delay is a random variable. In some other works, like [3], the local delay denotes the *mean* number of time slots required to successfully transmit a packet.

and variances of local delay we have derived, we explore the essential difference between the two MAC protocols. We also evaluate the optimal number of sub-bands for FHMA and the optimal transmit probability for ALOHA that minimize the mean local delay. The issue of optimizing the number of sub-bands was also considered in [10], where the optimal number of sub-bands is derived to maximize the number of concurrent transmissions. However, such outage-based framework used in [10] cannot capture the effects of correlated interference. In the last part of our work, we evaluate the mean delay-jitter tradeoff and the bounds on the tail probability of local delay, both of which are key topics for the system design.

Our results reveal that the mean local delay of the two protocols, FHMA and ALOHA, are the same when the number of sub-bands  $N$  in FHMA is equal to the reciprocal of the transmit probability  $p$  in ALOHA and the thermal noise is ignored. However, the variances of the local delay for the two protocols are rather different: when  $p = \frac{1}{N}$  and  $N \rightarrow \infty$ , the variance in FHMA tends to be a small constant, while in ALOHA the variance scales as  $\Theta(N^2)$ . Moreover, we calculate bounds on the complementary cumulative distribution function (ccdf) of the local delay when no MAC dynamic is employed, in this case, the distribution of the local delay has a heavy tail, which results in an infinite mean local delay. By employing the MAC randomness of either FHMA or ALOHA, the ccdf of local delay will decay fast, and the mean local delay will be finite. This observation reveals that even simply employing these kinds of randomness will greatly decrease the local delay.

### C. Related Works

Evaluating the interference correlation in wireless network is complicated: First, the spatial locations of interfering transmitters are uncertain, especially in ad hoc networks or heterogeneous networks with randomly scattered small cells. Second, closed-form results cannot be derived in the traditional grid model, e.g., hexagonal grid model for the cellular wireless network. Third, the fading and shadowing in the physical channel make the quantitative analysis difficult. Recently, the tools from stochastic geometry [11] have been used widely in modeling and analyzing of wireless communication systems;

see, e.g., [9], [12]–[14] and therein references. This mathematical framework permits the derivation of closed-form results for various system metrics and makes it possible to evaluate the interference correlation. A number of works considering the related problems are as follows. In [4] the authors calculated the correlation coefficient of interference under different assumptions of dependence, and in [5] the authors evaluated the spatio-temporal correlation coefficient of the interference and the joint probability of success in ALOHA networks. The framework for the analysis of the local delay was provided in [3] [6] [15] [16], where different scenarios were considered and it was observed that the mean local delay may be infinite under certain system parameters. In [17], a new model, which characterizes different degree of temporal dependence, was proposed to evaluate the local delay by using joint interference statistics. The work in [18] extended the results in [3] to the case of finite mobility. In [19], the optimal power control policies for different fading statistics were proposed to minimize the mean local delay. All the above works are based on the Poisson point process (PPP) model, while the work in [20] analyzed the local delay in clustered networks.

The remaining part of this paper is organized as follows. Section II describes the network model and the MAC schemes. Section III then establishes the main analytical results of this paper, including the means and variances of the local delay for FHMA and for ALOHA. Section IV evaluates the optimal number of sub-bands for FHMA and the optimal transmit probability for ALOHA that minimize the mean local delay. Section V evaluates the optimal SINR threshold that minimizes the mean local delay. Section VI presents the mean delay-jitter tradeoff and the bounds on the tail probability of local delay, and Section VI offers the concluding remarks.

## II. SYSTEM MODEL

### A. Network Model

To obtain the most essential features, we consider a simple yet meaningful model. We model the locations of the transmitters as a PPP  $\Phi = \{x_i\} \subset \mathbb{R}^d$  of intensity  $\lambda$ . In the analysis, we will condition on  $x_0 \in \Phi$ , where  $r_0 = |x_0|$  is the distance of this point to the origin. This transmitter is the typical transmitter under consideration, and we will assume

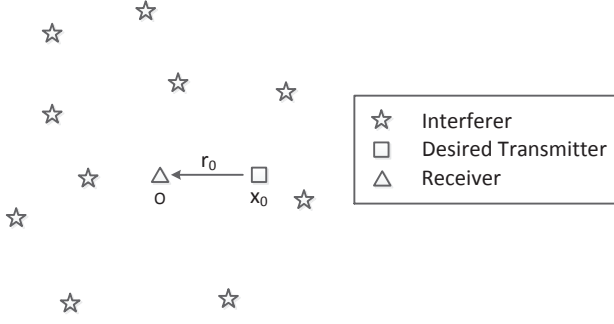


Fig. 1. Spatial distribution of different network entities.

that the corresponding receiver is located at the origin  $o$  (see Fig. 1).

We assume that time-division is introduced, in which case the time is divided into discrete slots with equal duration. Each transmission attempt occupies one time slot, and if a transmission fails in a certain time slot, a retransmission will be conducted. The local delay is defined as the number of time slots until a packet is successfully received [3] [6].

For the path loss, we assume that the power fading coefficients are spatially and temporally independent with exponential distribution of mean 1 (Rayleigh fading). Let  $\alpha$  be the path loss exponent and  $h_{k,x}$  be the fading coefficient between transmitter  $x$  and the considered receiver located at origin  $o$  in time slot  $k$ . We assume that all transmitters transmit at unit power. This constant power assumption is consistent with the bipolar network model, in which all link distances are identical, and we can set the constant power to 1 without loss of generality.

We assume that the SINR threshold model is applied. That is, for each degree of freedom (time-frequency resource block), as long as the SINR is above a threshold  $\theta$ , it can be successfully used for information transmission at spectral efficiency  $\log_2(1 + \theta)$  bits per second per Hz. We also assume that a packet of a fixed size needs exactly one time slot to be transmitted if it is allocated the entire frequency band  $W$  under SINR threshold  $\theta$  and successfully transmitted in that time slot. In that way, in the FHMA case if the entire frequency band is split into  $N$  sub-bands, a packet will need  $N$  successful time slots. Meanwhile, in the ALOHA case since the frequency band is not divided, each active transmission will make use of the entire frequency band; thus, only one successful time slot

is needed. Notice that the local delay is measured by the number of time slots. Since different system configurations may apply different durations of time slot, we should normalize the local delay so that the actual delays of different system configurations can be compared fairly. The duration of each time slot is proportional to  $\frac{1}{\log_2(1+\theta)}$  because the size of a packet is fixed and the spectral efficiency is proportional to  $\log_2(1 + \theta)$ . Therefore, when comparing the actual delays under different SINR thresholds  $\theta$ , we normalize the local delay by  $\frac{1}{\log_2(1+\theta)}$  as the metric.

In static or finitely mobile network, the locations of the transmitters during all time slots are deemed to be correlated, resulting in the interference correlation between different time slots. This type of correlation greatly decreases the successful probability for retransmissions if the first transmission attempt failed, thus leading to larger local delay. In order to reduce the effect of interference correlation and thus the local delay, we introduce two kinds of MAC randomness described as follows.

### B. FHMA

In the FHMA case, we assume that the total frequency band  $W$  is divided into  $N$  sub-bands and each transmitter chooses a sub-band uniformly randomly, independently of the location and the time slot (i.e., both spatially and temporally). Let  $s \in \mathbb{S} = \{1, 2, \dots, N\}$  be the sub-band index, and let  $\mathcal{S}_k(x) \in \mathbb{S}$  denote the index of the sub-band used by node  $x \in \Phi$  in time slot  $k$ .

With these notations, the interference at the typical receiver located at the origin  $o$  in time slot  $k$  is given by

$$I_k = \sum_{x \in \Phi \setminus \{x_0\}} h_{k,x} |x|^{-\alpha} \mathbf{1}(\mathcal{S}_k(x) = \mathcal{S}_k(x_0)), \quad (1)$$

where  $\mathbf{1}(\cdot)$  is the indicator function and  $|x|$  denotes the distance between  $x$  and the origin  $o$ . Note that the exclusion of  $x_0$  from the sum over the point process does not imply that  $x_0 \notin \Phi$ , but it ensures that when we condition on  $x_0 \in \Phi$ , the power received from this node is not counted as interference.

Besides reducing the interference and breaking the correlation, introducing FHMA has the additional benefit that the noise power decreases from  $WN_0$  to  $\frac{W}{N}N_0$ . By taking this noise scaling into

consideration, we obtain the SINR of the typical receiver in time slot  $k$  as

$$\text{SINR}_k = \frac{h_{k,x_0} r_0^{-\alpha}}{\frac{WN_0}{N} + \sum_{x \in \Phi \setminus \{x_0\}} h_{k,x} |x|^{-\alpha} \mathbf{1}(\mathcal{S}_k(x) = \mathcal{S}_k(x_0))}. \quad (2)$$

### C. ALOHA

In the ALOHA case, let  $\Phi_k$  be the transmitting set in time slot  $k$ . The interference at the typical receiver located at the origin  $o$  in time slot  $k$  is

$$I_k = \sum_{x \in \Phi \setminus \{x_0\}} h_{k,x} |x|^{-\alpha} \mathbf{1}(x \in \Phi_k). \quad (3)$$

Unlike FHMA, the noise scaling effect does not exist for ALOHA since the entire frequency band is used for each transmission. The SINR of the typical receiver in time slot  $k$  is

$$\text{SINR}_k = \frac{h_{k,x_0} r_0^{-\alpha}}{WN_0 + \sum_{x \in \Phi \setminus \{x_0\}} h_{k,x} |x|^{-\alpha} \mathbf{1}(x \in \Phi_k)}. \quad (4)$$

## III. MEAN AND VARIANCE OF THE LOCAL DELAY

In this section, we derive the means and variances of the local delay for FHMA and for ALOHA respectively.

### A. FHMA

1) *Mean local delay:* The following theorem gives the mean local delay in FHMA networks.

*Theorem 1:* In FHMA with  $N$  sub-bands, the mean local delay is

$$D(N) = N \exp \left( \frac{A}{(N-1)^{1-\delta} N^\delta} + \frac{B}{N} \right), \quad (5)$$

where  $A = \lambda c_d r_0^d \theta^\delta C(\delta)$ ,  $B = \theta r_0^\alpha W N_0$ ,  $\delta = d/\alpha$ ,  $C(\delta) = 1/\text{sinc}(\delta)$ , and  $c_d = |b(o, 1)|$  is the volume of the  $d$ -dimensional unit ball<sup>2</sup>.

*Proof:* Let  $\mathcal{C}_\Phi$  be the event that a transmission succeeds conditioned on the PPP  $\Phi$ . The probability for successful transmission given  $\Phi$  is the same for each time slot. Our analysis below is conditioned on  $\Phi$  having a point at  $x_0$ . This means that the probability measure of the point process is the Palm

<sup>2</sup>Since the equation (5) has implied that  $D(1) = \infty$ , without loss of generality, we can regard the domain of  $D(N)$  as  $N \geq 1$  with  $D(1) = \infty$ .

probability  $\mathbb{P}^{x_0}$  (see Ch. 8 in [9]). Correspondingly, the expectation, denoted by  $\mathbb{E}^{x_0}$ , is taken with respect to the measure  $\mathbb{P}^{x_0}$ . With this notation, by setting the SINR threshold to be  $\theta$ , we denote the probability of successful transmission conditioned on  $\Phi$  as  $\mathbb{P}^{x_0}(\mathcal{C}_\Phi) = \mathbb{P}^{x_0}(\text{SINR}_k > \theta \mid \Phi)$ , which can be evaluated as

$$\begin{aligned} \mathbb{P}^{x_0}(\mathcal{C}_\Phi) &= \mathbb{P}^{x_0}(\text{SINR}_k > \theta \mid \Phi) \\ &= \mathbb{P}^{x_0} \left( h_{k,x_0} r_0^{-\alpha} > \theta \left( \frac{W}{N} N_0 + I_k \right) \mid \Phi \right) \\ &\stackrel{(a)}{=} \mathbb{E}^{x_0} \left( \exp \left( -\theta r_0^\alpha \left( \frac{W}{N} N_0 + I_k \right) \right) \mid \Phi \right) \\ &= \mathbb{E}^{x_0} \left( \exp \left( -\theta r_0^\alpha \frac{W}{N} N_0 - \right. \right. \\ &\quad \left. \left. \sum_{x \in \Phi \setminus \{x_0\}} \theta r_0^\alpha h_{k,x} |x|^{-\alpha} \mathbf{1}(\mathcal{S}_k(x) = \mathcal{S}_k(x_0)) \right) \mid \Phi \right) \\ &= \exp \left( -\frac{\theta r_0^\alpha W N_0}{N} \right) \\ &\quad \prod_{x \in \Phi \setminus \{x_0\}} \mathbb{E}^{x_0} \left( \exp \left( -\theta r_0^\alpha h_{k,x} |x|^{-\alpha} \mathbf{1}(\mathcal{S}_k(x) = \mathcal{S}_k(x_0)) \right) \mid \Phi \right) \\ &= \exp \left( -\frac{\theta r_0^\alpha W N_0}{N} \right) \\ &\quad \prod_{x \in \Phi \setminus \{x_0\}} \left( \frac{1}{N} \mathbb{E}^{x_0} \left( \exp \left( -\theta r_0^\alpha h_{k,x} |x|^{-\alpha} \right) \mid \Phi \right) + \frac{N-1}{N} \right) \\ &\stackrel{(b)}{=} \exp \left( -\frac{\theta r_0^\alpha W N_0}{N} \right) \prod_{x \in \Phi \setminus \{x_0\}} \left( \frac{1}{N} \frac{1}{1 + \theta r_0^\alpha |x|^{-\alpha}} + \frac{N-1}{N} \right). \end{aligned} \quad (6)$$

In steps (a) and (b) of the derivation above, we have applied the property that the fading coefficients  $h_{k,x}$  are i.i.d. random variables with exponential distribution of unit mean. The number of time slots needed until a successful time slot appears, denoted by  $\Delta$ , is a random variable called *delay till success* (DTS) [19]. Conditioned upon  $\Phi$ , the success events in different time slots are independent with probability  $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$ ; therefore, the DTS with given  $\Phi$ , denoted by  $\Delta_\Phi$ , is a random variable with geometric distribution given by

$$\mathbb{P}^{x_0}(\Delta_\Phi = k) = (1 - \mathbb{P}^{x_0}(\mathcal{C}_\Phi))^{k-1} \mathbb{P}^{x_0}(\mathcal{C}_\Phi). \quad (7)$$

The conditional expectation of  $\Delta_\Phi$  is taken w.r.t. the fading and the MAC, given by  $\mathbb{E}^{x_0}(\Delta_\Phi) = 1/\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$ . Noticing that a packet will need  $N$  successful time slots to finish transmission in FHMA,

the mean local delay can be evaluated as

$$\begin{aligned}
D(N) &= N\mathbb{E}^{x_0}(\Delta) \\
&= N\mathbb{E}_{\Phi}^{x_0}(\mathbb{E}^{x_0}(\Delta_{\Phi})) \\
&= N\mathbb{E}_{\Phi}^{x_0}\left(\frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})}\right) \\
&\stackrel{(a)}{=} N\exp\left(\frac{\theta r_0^{\alpha} W N_0}{N}\right) \\
&\quad \mathbb{E}_{\Phi}^{x_0}\left(\frac{1}{\prod_{x \in \Phi \setminus \{x_0\}} \left(\frac{1}{N} \frac{1}{1 + \theta r_0^{\alpha} |x|^{-\alpha}} + \frac{N-1}{N}\right)}\right) \\
&= N\exp\left(\frac{\theta r_0^{\alpha} W N_0}{N}\right) \\
&\quad \mathbb{E}_{\Phi}^{x_0}\left(\prod_{x \in \Phi \setminus \{x_0\}} \frac{1}{\frac{1}{N} \frac{1}{1 + \theta r_0^{\alpha} |x|^{-\alpha}} + \frac{N-1}{N}}\right). \quad (8)
\end{aligned}$$

where (a) follows from (6). By applying the probability generating functional (PGFL) of the PPP, we obtain

$$\begin{aligned}
D(N) &= N\exp\left(\frac{\theta r_0^{\alpha} W N_0}{N}\right) \\
&\quad \exp\left(-\lambda \int_{\mathbb{R}^d} \left(1 - \frac{1}{\frac{1}{N} \frac{1}{1 + \theta r_0^{\alpha} |x|^{-\alpha}} + \frac{N-1}{N}}\right) dx\right) \\
&= N\exp\left(\frac{\theta r_0^{\alpha} W N_0}{N} + \lambda c_d d \int_0^{\infty} \frac{r^{d-1}}{\frac{N}{\theta r_0^{\alpha}} r^{\alpha} + N-1} dr\right) \\
&= N\exp\left(\frac{\lambda c_d r_0^d \theta^{\delta} C(\delta)}{(N-1)^{1-\delta} N^{\delta}} + \frac{\theta r_0^{\alpha} W N_0}{N}\right). \quad (9)
\end{aligned}$$

where  $\delta = d/\alpha$ ,  $C(\delta)$  is given by  $C(\delta) = \Gamma(1+\delta)\Gamma(1-\delta) = \frac{1}{\text{sinc}(\delta)}$ , and  $c_d = |b(o, 1)|$  is the volume of the  $d$ -dimensional unit ball. ■

The result in Theorem 1 is closed-form and easy to evaluate and analyze. The value of  $A$  is determined by the interference and that of  $B$  is due to the thermal noise. From (5), we have

$$\begin{aligned}
D(N) &= N\exp\left(A\left(1 - \frac{1}{N}\right)^{\delta-1} \frac{1}{N} + \frac{B}{N}\right) \\
&= N\exp\left(A\left(1 - \frac{\delta-1}{N} + O\left(\frac{1}{N^2}\right)\right) \frac{1}{N} + \frac{B}{N}\right) \\
&= N\exp\left(\frac{A+B}{N} + O\left(\frac{1}{N^2}\right)\right) \\
&= N + A + B + O\left(\frac{1}{N}\right). \quad (10)
\end{aligned}$$

The result shows that when  $N$  is large, the mean local delay increases linearly with  $N$ . Since  $D(1)$

is infinity, there exists an optimal number of sub-bands  $N_{\text{opt}}$  that minimizes the mean local delay. Inspecting  $D(N)$ , we see that there are two effects by splitting the entire frequency band into  $N$  sub-bands: first, the mean local delay  $D(N)$  tends to decrease due to the smaller correlation; second,  $D(N)$  tends to increase since the number of time slots needed becomes  $N$  times larger. In view of this, we introduce two regimes, *correlation-limited* regime and *bandwidth-limited* regime. For  $N < N_{\text{opt}}$ , the second effect outweighs the first one, and the network operates in the correlation-limited regime. For  $N > N_{\text{opt}}$ , it is the opposite and the network operates in the bandwidth-limited regime.

In the above, we have derived the results for the case where the frequency allocation is dynamic (i.e., the sub-bands are allocated randomly and independently in each time slot). Alternatively, one could consider the case where the frequency allocation is also static over time. That case is exactly the same as the case where no frequency splitting is applied, with the only difference that the intensity of the interfering transmitters is  $\lambda/N$ . The mean local delay in that case is also infinite. This fact explains that even though frequency splitting is introduced, if the sub-bands are not reallocated randomly in each time slot, the mean local delay will still be infinite. This is a nontrivial observation since it reveals that the reduction of the mean local delay by introducing FHMA does not come from reducing the interference or the thermal noise, but mainly comes from reducing the *interference correlation*.

Based on Theorem 1, we show how the normalized mean local delay  $\frac{D(N)}{\log_2(1+\theta)}$  varies with  $N$  numerically. As for the parameters, we ignore the thermal noise ( $N_0 = 0$ ) and set the intensity of transmitters as  $\lambda = 0.0001\text{m}^{-2}$  by default, which means that the coverage area of each transmitter is  $10000\text{m}^2$  on average, reasonable for a typical deployment of WLAN. The path loss exponent is set as  $\alpha = 4$  by default, and the distance between the receiver and the typical desired transmitter is  $r = 50\text{m}$ . Let  $\theta$  be the outage threshold for SINR. The relationship between  $\frac{D(N)}{\log_2(1+\theta)}$  and  $N$  is depicted in Fig. 2.

By changing the values of  $\alpha$  and  $\lambda$  respectively, we get the curves in Fig. 2. Comparing the curves in Fig. 2(a) with those in Fig. 2(b) and Fig. 2(c), we observe that the optimal number of sub-bands increases when  $\alpha$  decreases or when  $\lambda$  increases.

This observation is consistent with the intuition: Smaller  $\alpha$  implies that the signal strength decays more slowly with distance, and larger  $\lambda$  implies that more transmitters exist in the same region, so in both cases more interference is created. Therefore, the entire frequency band should be divided into more sub-bands, namely larger  $N_{\text{opt}}$ , to reduce the interference.

2) *Variance of the local delay*: The mean local delay discussed above has characterized the mean number of time slots needed until a packet is successfully transmitted. In order to better understand the distribution of the local delay, we also derive its variance. The following theorem gives the variance of the local delay for FHMA.

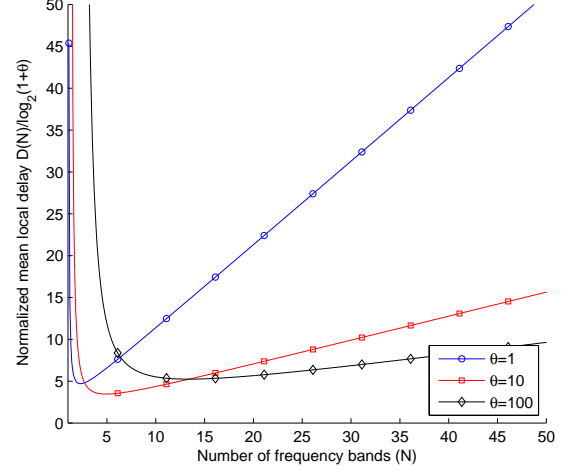
*Theorem 2*: In FHMA with  $N$  sub-bands, the variance of the local delay is

$$V(N) = N(N+1) \exp\left(\frac{(2N-1-\delta)A}{N^\delta(N-1)^{2-\delta}} + \frac{2B}{N}\right) - D(N) - D^2(N). \quad (11)$$

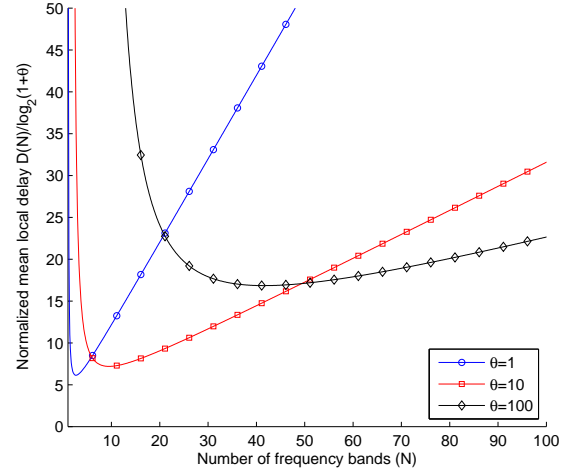
*Proof*: In order to transmit a packet in FHMA,  $N$  successful transmissions are needed. Letting  $\Delta_i$  ( $1 \leq i \leq N$ ) be the DTS of the  $i$ th transmission, we get the local delay of a packet as  $\sum_{i=1}^N \Delta_i$ . For  $1 \leq i, j \leq N$  and  $i \neq j$ ,  $\Delta_i$  and  $\Delta_j$  are dependent because the interference of the  $i$ th transmission and that of the  $j$ th transmission are correlated. However, if we condition on  $\Phi$ ,  $\{\Delta_i\}$  are i.i.d. random variables with geometric distribution given by (7). With these notations, we obtain the variance of the local delay as

$$\begin{aligned} V(N) &= \mathbb{E}^{x_0} \left( \left( \sum_{i=1}^N \Delta_i \right)^2 \right) - \left( \mathbb{E}^{x_0} \left( \sum_{i=1}^N \Delta_i \right) \right)^2 \\ &= \mathbb{E}^{x_0} \left( \sum_{i=1}^N \Delta_i^2 + \sum_{\substack{i,j=1 \\ i \neq j}}^N 2\Delta_i \Delta_j \right) - \left( \sum_{i=1}^N \mathbb{E}^{x_0}(\Delta_i) \right)^2 \\ &\stackrel{(a)}{=} \sum_{i=1}^N \mathbb{E}^{x_0}(\Delta_i^2) + \sum_{\substack{i,j=1 \\ i \neq j}}^N 2\mathbb{E}^{x_0}(\Delta_i \Delta_j) - D^2(N), \end{aligned}$$

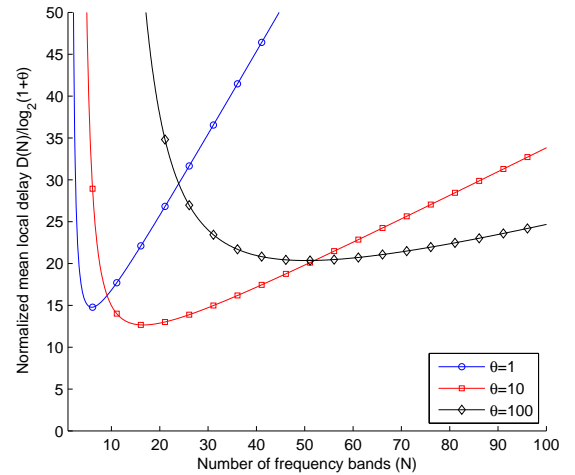
where (a) follows from the definition of the mean local delay. By applying the total expectation for-



(a)  $\lambda = 0.0001$  and  $\alpha = 4$ .



(b)  $\lambda = 0.0001$  and  $\alpha = 3$ .



(c)  $\lambda = 0.0004$  and  $\alpha = 4$ .

Fig. 2. The normalized mean local delay  $\frac{D(N)}{\log_2(1+\theta)}$  as a function of the number of sub-bands  $N$ , when  $d = 2$ ,  $r = 50\text{m}$ , and thermal noise ignored.

mula, we have

$$\begin{aligned}
V(N) &= \sum_{i=1}^N \mathbb{E}_{\Phi}^{x_0} (\mathbb{E}^{x_0} (\Delta_i^2 | \Phi)) \\
&\quad + \sum_{\substack{i,j=1 \\ i \neq j}}^N 2\mathbb{E}_{\Phi}^{x_0} (\mathbb{E}^{x_0} (\Delta_i \Delta_j | \Phi)) - D^2(N) \\
&\stackrel{(b)}{=} N\mathbb{E}_{\Phi}^{x_0} \left( \frac{2 - \mathbb{P}^{x_0}(\mathcal{C}_{\Phi})}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) \\
&\quad + N(N-1)\mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) - D^2(N) \\
&= N(N+1)\mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) \\
&\quad - N\mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})} \right) - D^2(N) \\
&= N(N+1)\mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) \\
&\quad - D(N) - D^2(N), \tag{12}
\end{aligned}$$

where (b) follows from the second moment of the geometrically distributed random variable.

From (6), we have

$$\begin{aligned}
&\mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) \\
&= \exp \left( \frac{2\theta r_0^{\alpha} W N_0}{N} \right) \\
&\quad \mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{\prod_{x \in \Phi \setminus \{x_0\}} \left( \frac{1}{N} \frac{1}{1 + \theta r_0^{\alpha} |x|^{-\alpha}} + \frac{N-1}{N} \right)^2} \right) \\
&\stackrel{(c)}{=} \exp \left( \frac{2\theta r_0^{\alpha} W N_0}{N} \right. \\
&\quad \left. - \lambda \int_{\mathbb{R}^d} \left( 1 - \frac{1}{\left( \frac{1}{N} \frac{1}{1 + \theta r_0^{\alpha} |x|^{-\alpha}} + \frac{N-1}{N} \right)^2} \right) dx \right) \\
&= \exp \left( \frac{2\theta r_0^{\alpha} W N_0}{N} \right. \\
&\quad \left. - \lambda c_d \int_0^{\infty} \left( 1 - \frac{N^2(1 + \theta r_0^{\alpha} r^{-\alpha})^2}{(N + (N-1)\theta r_0^{\alpha} r^{-\alpha})^2} \right) r^{d-1} dr \right) \\
&= \exp \left( \frac{2\theta r_0^{\alpha} W N_0}{N} + \frac{\lambda c_d r_0^d \theta^{\delta} C(\delta)(2N-1-\delta)}{N^{\delta}(N-1)^{2-\delta}} \right), \tag{13}
\end{aligned}$$

where (c) follows by applying the PGFL of the PPP. Plugging (13) into (12), we get the variance of the local delay as in Theorem 2. ■

## B. ALOHA

The fundamental difference between FHMA and ALOHA is that if a packet is to be transmitted during a time slot, in FHMA the packet will be surely transmitted by randomly choosing a sub-band, while in ALOHA the packet will only be transmitted with a given probability. Similar to the analysis of FHMA, we also assume that a packet needs exactly one time slot if it is allocated the entire frequency band  $W$  under SINR threshold  $\theta$  and successfully transmitted in that time slot. We assume that each node transmits with probability  $p$  in each time slot and if it transmits, it will make use of the entire frequency band. In that way, only one successful time slot is needed to transmit a packet, and the local delay is the DTS of one transmission, denoted by  $\Delta$ .

1) *Mean local delay:* The following theorem gives the mean local delay for ALOHA.

*Theorem 3:* In ALOHA with transmit probability  $p$ , the mean local delay is

$$\tilde{D}(p) = \frac{1}{p} \exp \left( \frac{pA}{(1-p)^{1-\delta}} + B \right). \tag{14}$$

*Proof:* In each time slot, a packet will be transmitted with probability  $p$  and the transmission will be successful with probability  $\mathbb{P}^{x_0}(\text{SINR}_k > \theta | \Phi)$  conditioned upon  $\Phi$ . Therefore, similar to the derivation of (6), the probability for successfully transmitting a packet conditioned upon  $\Phi$  in a time slot is

$$\begin{aligned}
&\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}) \stackrel{(a)}{=} p\mathbb{P}^{x_0}(\text{SINR}_k > \theta | \Phi) \\
&= p\mathbb{P}^{x_0}(h_{k,x_0}r_0^{-\alpha} > \theta(WN_0 + I_k) | \Phi) \\
&\stackrel{(b)}{=} p\mathbb{E}^{x_0}(\exp(-\theta r_0^{\alpha}(WN_0 + I_k)) | \Phi) \\
&= p\mathbb{E}^{x_0} \left( \exp \left( -\theta r_0^{\alpha} W N_0 \right. \right. \\
&\quad \left. \left. - \sum_{x \in \Phi \setminus \{x_0\}} \theta r_0^{\alpha} h_{k,x} |x|^{-\alpha} \mathbf{1}(x \in \Phi_k) \right) | \Phi \right) \\
&= p \exp(-\theta r_0^{\alpha} W N_0) \\
&\quad \prod_{x \in \Phi \setminus \{x_0\}} \mathbb{E}^{x_0}(\exp(-\theta r_0^{\alpha} h_{k,x} |x|^{-\alpha} \mathbf{1}(x \in \Phi_k)) | \Phi) \\
&= p \exp(-\theta r_0^{\alpha} W N_0) \\
&\quad \prod_{x \in \Phi \setminus \{x_0\}} \left( p\mathbb{E}^{x_0}(\exp(-\theta r_0^{\alpha} h_{k,x} |x|^{-\alpha}) | \Phi) + 1 - p \right) \\
&\stackrel{(c)}{=} p \exp(-\theta r_0^{\alpha} W N_0) \prod_{x \in \Phi \setminus \{x_0\}} \left( \frac{p}{1 + \theta r_0^{\alpha} |x|^{-\alpha}} + 1 - p \right). \tag{15}
\end{aligned}$$



where (a) is because a transmission occurs with probability  $p$ , and (b) and (c) follows because the fading coefficients  $h_{k,x}$  are i.i.d. random variables with exponential distribution of unit mean. Then, the mean local delay for ALOHA is given by

$$\begin{aligned}\tilde{D}(p) &= \mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})} \right) \\ &= \frac{1}{p} \exp \left( \theta r_0^{\alpha} W N_0 \right. \\ &\quad \left. - \lambda \int_{\mathbb{R}^d} \left( 1 - \frac{1}{p \frac{1}{1+\theta r_0^{\alpha}|x|^{-\alpha}} + 1 - p} \right) dx \right) \\ &= \frac{1}{p} \exp \left( \frac{\lambda c_d r_0^d \theta^{\delta} C(\delta) p}{(1-p)^{1-\delta}} + \theta r_0^{\alpha} W N_0 \right).\end{aligned}\quad (16)$$

Applying the definition of  $A$  and  $B$  in Theorem 1, we obtain the results in Theorem 3. ■

2) *Variance of the local delay:* The variance of the local delay in ALOHA is given by the following theorem.

*Theorem 4:* In ALOHA with transmit probability  $p$ , the variance of the local delay is

$$\begin{aligned}\tilde{V}(p) &= \frac{2}{p^2} \exp \left( \frac{(2-p-\delta p)pA}{(1-p)^{2-\delta}} + 2B \right) \\ &\quad - \tilde{D}(p) - \tilde{D}^2(p).\end{aligned}\quad (17)$$

*Proof:* In the ALOHA case, in order to transmit a packet, one successful transmission is needed. The variance of local delay for ALOHA is thus

$$\begin{aligned}\tilde{V}(p) &= \mathbb{E}^{x_0} (\Delta^2) - (\mathbb{E}^{x_0} (\Delta))^2 \\ &\stackrel{(a)}{=} \mathbb{E}_{\Phi}^{x_0} (\mathbb{E}^{x_0} (\Delta^2 | \Phi)) - \tilde{D}^2(p) \\ &\stackrel{(b)}{=} \mathbb{E}_{\Phi}^{x_0} \left( \frac{2 - \mathbb{P}^{x_0}(\mathcal{C}_{\Phi})}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) - \tilde{D}^2(p) \\ &= 2\mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) - \tilde{D}(p) - \tilde{D}^2(p),\end{aligned}\quad (18)$$

where (a) follows from the total expectation formula, and (b) follows from the second moment of the geometrically distributed random variable. From

(15), we have

$$\begin{aligned}\mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{(\mathbb{P}^{x_0}(\mathcal{C}_{\Phi}))^2} \right) &= p^{-2} \exp (2\theta r_0^{\alpha} W N_0) \\ &\quad \mathbb{E}_{\Phi}^{x_0} \left( \frac{1}{\prod_{x \in \Phi \setminus \{x_0\}} \left( p \frac{1}{1+\theta r_0^{\alpha}|x|^{-\alpha}} + 1 - p \right)^2} \right) \\ &\stackrel{(c)}{=} p^{-2} \exp \left( 2\theta r_0^{\alpha} W N_0 \right. \\ &\quad \left. - \lambda \int_{\mathbb{R}^d} \left( 1 - \frac{1}{\left( p \frac{1}{1+\theta r_0^{\alpha}|x|^{-\alpha}} + 1 - p \right)^2} \right) dx \right) \\ &= \frac{1}{p^2} \exp \left( 2\theta r_0^{\alpha} W N_0 + \frac{\lambda c_d r_0^d \theta^{\delta} C(\delta) (2-p-\delta p)p}{(1-p)^{2-\delta}} \right),\end{aligned}\quad (19)$$

where (c) follows from the probability generating functional (PGFL) of the PPP. Plugging (19) into (18) and applying the definition of  $A$  and  $B$  in Theorem 1, we get the variance of the local delay in Theorem 4. ■

### C. Comparison Between FHMA and ALOHA

1) *Mean local delay:* The mean local delay in FHMA is given by  $D(N)$  in (5), and that in ALOHA is given by  $\tilde{D}(p)$  in (14). In ALOHA, if the transmit probability  $p$  is set as  $\frac{1}{N}$ , by comparing  $\tilde{D}(\frac{1}{N})$  to the result of FHMA,  $D(N)$  given in (5), we observe that the only difference lies in the thermal noise term. In FHMA, the entire frequency band is divided into a number of sub-bands, thus reducing the noise power. However, with ALOHA, the noise scaling effect does not exist since the entire frequency band is used for transmission. The mean local delays of the two schemes are the same if we ignore the thermal noise term and set  $p = \frac{1}{N}$ , which is an instance of the time-frequency duality.

2) *Variance of the local delay:* Comparing (11) and (17), we observe that even if the noise is ignored and the transmit probability is set as  $p = \frac{1}{N}$  in ALOHA, there is still a significant difference between the variances of the two schemes when  $N > 1$ : in FHMA, a factor  $N(N+1)$  exists in the first term, while in ALOHA the factor is  $2N^2$ . When  $N > 1$ , we have  $V(N) < \tilde{V}(\frac{1}{N})$  and this illustrates that the variance of the local delay for FHMA is less than that for ALOHA (see Fig. 3). We further observe from Fig. 3 that when  $N \rightarrow \infty$ , the variance

for FHMA stabilizes at a typically small value, while for ALOHA, the variance increases quickly with  $N$ . To understand the limiting characteristics quantitatively, we evaluate how the variances of the local delay scale with  $N$  in the following proposition.

*Proposition 1:* For FHMA with number of sub-bands  $N > 1$ , the variance of local delay is

$$V(N) = (2 - \delta)A + B + O\left(\frac{1}{N}\right) = \Theta(1). \quad (20)$$

For ALOHA with transmit probability  $p = \frac{1}{N} < 1$ , the variance of the local delay is

$$\tilde{V}\left(\frac{1}{N}\right) = \Theta(N^2). \quad (21)$$

*Proof:* For FHMA with number of sub-bands  $N > 1$ , from (5), we have

$$\begin{aligned} D(N) &= N \exp\left(\frac{A+B}{N} + \frac{(1-\delta)A}{N^2} + O\left(\frac{1}{N^3}\right)\right) \\ &= N + (A+B) \\ &\quad + \left(\frac{(A+B)^2}{2} + (1-\delta)A\right)\frac{1}{N} + O\left(\frac{1}{N^2}\right). \end{aligned} \quad (22)$$

Then,  $D^2(N)$  is given by

$$\begin{aligned} D^2(N) &= N^2 + 2(A+B)N + 2(A+B)^2 \\ &\quad + 2(1-\delta)A + O\left(\frac{1}{N}\right). \end{aligned} \quad (23)$$

From (11), we have the variance of the local delay as

$$\begin{aligned} V(N) &= N(N+1) \exp\left(\frac{2B}{N} + \frac{2A}{N} - \frac{3(\delta-1)A}{N^2} + O\left(\frac{1}{N^3}\right)\right) - D(N) - D^2(N) \\ &= N^2 + (2A+2B+1)N + 2(A+B) \\ &\quad + 2(A+B)^2 - 3(\delta-1)A \\ &\quad - D(N) - D^2(N) + O\left(\frac{1}{N}\right). \end{aligned} \quad (24)$$

Plugging (22) and (23) into (24), we get the result in (20). The derivations of the limiting for ALOHA is similar, and we omit the details of that proof. ■

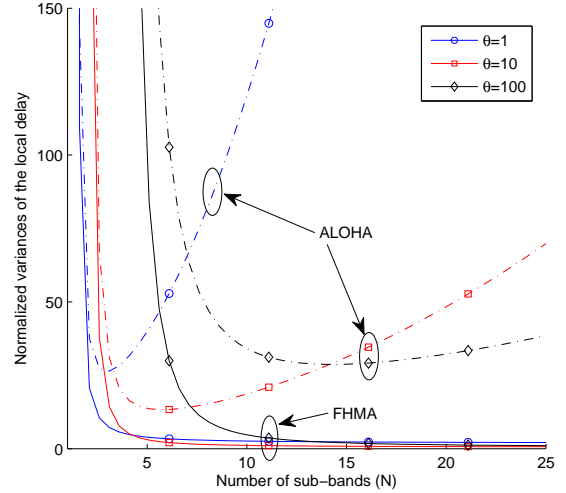


Fig. 3. Normalized variances of the local delay for FHMA,  $\frac{V(N)}{(\log_2(1+\theta))^2}$ , and for ALOHA,  $\frac{\tilde{V}(\frac{1}{N})}{(\log_2(1+\theta))^2}$ , as a function of the number of sub-bands  $N$ , when  $d = 2$ ,  $\lambda = 0.0001\text{m}^{-2}$ ,  $\alpha = 4$ ,  $r = 50\text{m}$ , and thermal noise ignored.

#### D. Finiteness of the Mean Local Delay

To understand why the mean local delay goes to infinity when  $N$  is set to one, let us consider the expression for the mean local delay  $D = N\mathbb{E}_{\Phi}^{x_0}\left(\frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})}\right)$  in FHMA, where  $\frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})}$  is a random variable with support set  $(1, +\infty)$  because it is the reciprocal of the successful transmit probability conditioned upon the PPP  $\Phi$ . When  $N = 1$ , the expectation  $\mathbb{E}_{\Phi}^{x_0}\left(\frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})}\right)$  is infinity because the ccdf of  $\frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})}$  has a heavy tail. To show the heavy tail behavior, let us derive a lower bound for the ccdf of the local delay when  $N = 1$ . Ignoring the thermal noise term in (6), for  $N = 1$  and any  $t \in (1, +\infty)$ , we have

$$\begin{aligned} \mathbb{P}^{x_0}\left(\frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_{\Phi})} > t\right) &= \mathbb{P}^{x_0}\left(\prod_{x \in \Phi \setminus \{x_0\}} (1 + \theta r_0^{\alpha} |x|^{-\alpha}) > t\right) \\ &> \mathbb{P}^{x_0}\left(1 + \theta r_0^{\alpha} |x_{\min}|^{-\alpha} > t\right) \\ &> \mathbb{P}^{x_0}\left(\theta r_0^{\alpha} |x_{\min}|^{-\alpha} > t\right) \\ &= \mathbb{P}^{x_0}\left(|x_{\min}| < \theta^{\frac{1}{\alpha}} t^{-\frac{1}{\alpha}} r_0\right), \end{aligned} \quad (25)$$

where  $x_{\min} = \arg\min_{x \in \Phi \setminus \{x_0\}} |x|$  is the nearest interfering transmitter to the receiver. The distance between the receiver and its nearest interfering transmitter has cumulative distribution function (cdf) as  $b(r) = 1 - \exp(-c_d \lambda r^d)$ . Substituting the cdf into

(25) and letting  $\delta = \frac{2}{\alpha}$ , we get

$$\mathbb{P}^{x_0} \left( \frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_\Phi)} > t \right) > 1 - \exp(-c_d \lambda \theta^\delta t^{-\delta} r_0^d) \quad (26)$$

$$\sim \frac{c_d \lambda \theta^\delta r_0^d}{t^\delta}, \quad t \rightarrow \infty. \quad (27)$$

The function  $g(t) = 1 - \exp(-C_0 t^{-\delta})$ , where  $C_0 = c_d \lambda \theta^\delta r_0^d$ , gives a lower bound for the ccdf of the local delay when  $N = 1$  (see Fig. 4).

By the identity of  $\mathbb{E}(X) = \int_0^\infty \mathbb{P}(X > t) dt$  for any non-negative random variable  $X$  and the inequality  $e^{-x} < 1 - x + \frac{x^2}{2}$  for  $x > 0$ , we have

$$\begin{aligned} \mathbb{E}_\Phi^{x_0} \left( \frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_\Phi)} \right) &= \int_0^\infty \left( \mathbb{P}^{x_0} \left( \frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_\Phi)} > t \right) \right) dt \\ &= 1 + \int_1^\infty \left( \mathbb{P}^{x_0} \left( \frac{1}{\mathbb{P}^{x_0}(\mathcal{C}_\Phi)} > t \right) \right) dt \\ &> 1 + \int_1^\infty (1 - \exp(-C_0 t^{-\delta})) dt \\ &> 1 + \int_1^\infty \left( C_0 t^{-\delta} - \frac{1}{2} C_0^2 t^{-2\delta} \right) dt \\ &= \infty. \end{aligned} \quad (28)$$

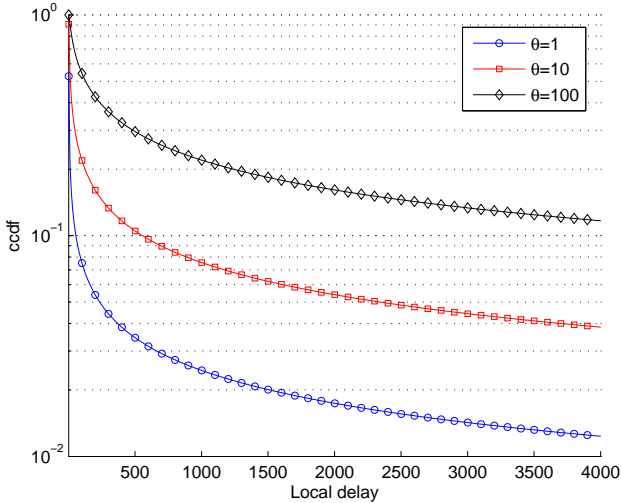


Fig. 4. Lower bound for the ccdf of the local delay, given by (26), when  $N = 1$  in the 2-dimensional case ( $d = 2$ ). The intensity of transmitters is  $\lambda = 0.0001 \text{m}^{-2}$  and the path loss exponent is  $\alpha = 4$ .

When FHMA is applied, there is an additional term  $\frac{N-1}{N}$  in the success probability given by (6), which prevents  $\mathbb{P}^{x_0}(\mathcal{C}_\Phi)$  from getting too small when  $|x|$  approaches zero. It can be interpreted intuitively that, although there are some interfering transmitters very close to the receiver, the application of FHMA guarantees that there is always a

relatively large probability that those transmitters do not simultaneously cause interference to the receiver.

#### IV. OPTIMAL PARAMETERS TO MINIMIZE MEAN LOCAL DELAY

In this section, we analyze the optimal number of sub-bands in FHMA and optimal transmit probability in ALOHA to minimize the mean local delay. Deriving the optimal parameters is difficult, and the result may not be compact; thus, we resort to deriving tight bounds for the optimal values.

##### A. FHMA

In the derivation, we assume that  $N$  is continuous and that the actual optimal number is a nearby integer. The following theorem gives the bounds of the optimal number of sub-bands.

*Theorem 5:* The bounds of the optimal number of sub-bands that minimizes the mean local delay are given by

$$N_{\text{opt}} \in (t_0, t_0 + 2), t_0 = \lambda c_d r_0^d \theta^\delta C(\delta) + \theta r_0^\alpha W N_0. \quad (29)$$

The actual optimal number is an integer, and the bounds are as follows:

$$N_{\text{opt}} \in [\lfloor t_0 \rfloor, \lceil t_0 \rceil + 2], t_0 = \lambda c_d r_0^d \theta^\delta C(\delta) + \theta r_0^\alpha W N_0. \quad (30)$$

*Proof:* Based on the result of (5), we get the derivative of the mean local delay  $D'(N)$  when  $N > 1$  as follows

$$\begin{aligned} D'(N) &= f(N) \exp(\lambda c_d r_0^d \theta^\delta C(\delta) (N-1)^{\delta-1} N^{-\delta} \\ &\quad + \theta r_0^\alpha W N_0 N^{-1}), \end{aligned} \quad (31)$$

where

$$\begin{aligned} f(N) &= 1 - \frac{1}{N} \left( \lambda c_d r_0^d \theta^\delta C(\delta) \left( \frac{N}{N-1} \right)^{1-\delta} \frac{N-\delta}{N-1} \right. \\ &\quad \left. + \theta r_0^\alpha W N_0 \right). \end{aligned} \quad (32)$$

We observe that  $f(N)$  is strictly monotonically increasing in  $N$ ; this means that there is only one optimal value  $N_{\text{opt}}$  that satisfies  $D'(N_{\text{opt}}) = 0$ ,

which is given by  $f(N_{\text{opt}}) = 0$ . From  $f(N_{\text{opt}}) = 0$ , we get

$$N_{\text{opt}} = \lambda c_d r_0^d \theta^\delta C(\delta) \left( \frac{N_{\text{opt}}}{N_{\text{opt}} - 1} \right)^{2-\delta} \frac{N_{\text{opt}} - \delta}{N_{\text{opt}}} + \theta r_0^\alpha W N_0. \quad (33)$$

Since  $N_{\text{opt}}/(N_{\text{opt}} - 1) > 1$  and  $0 < \delta < 1$ , we have

$$\begin{aligned} N_{\text{opt}} &> \lambda c_d r_0^d \theta^\delta C(\delta) \left( \frac{N_{\text{opt}}}{N_{\text{opt}} - 1} \right)^{2-\delta} \frac{N_{\text{opt}} - 1}{N_{\text{opt}}} \\ &\quad + \theta r_0^\alpha W N_0 \\ &> \lambda c_d r_0^d \theta^\delta C(\delta) + \theta r_0^\alpha W N_0, \end{aligned} \quad (34)$$

This gives a lower bound for the optimal value  $N_{\text{opt}}$ . Next we derive an upper bound for  $N_{\text{opt}}$ . From (33), we have

$$\begin{aligned} N_{\text{opt}} &< \lambda c_d r_0^d \theta^\delta C(\delta) \left( \frac{N_{\text{opt}}}{N_{\text{opt}} - 1} \right)^2 + \theta r_0^\alpha W N_0 \\ &< (\lambda c_d r_0^d \theta^\delta C(\delta) + \theta r_0^\alpha W N_0) \left( \frac{N_{\text{opt}}}{N_{\text{opt}} - 1} \right)^2. \end{aligned}$$

Then, we have

$$\frac{(N_{\text{opt}} - 1)^2}{N_{\text{opt}}} < \lambda c_d r_0^d \theta^\delta C(\delta) + \theta r_0^\alpha W N_0.$$

$$N_{\text{opt}} - 2 + \frac{1}{N_{\text{opt}}} < \lambda c_d r_0^d \theta^\delta C(\delta) + \theta r_0^\alpha W N_0.$$

$$N_{\text{opt}} < \lambda c_d r_0^d \theta^\delta C(\delta) + \theta r_0^\alpha W N_0 + 2. \quad (35)$$

Combining (34) and (35), we get the bounds of  $N_{\text{opt}}$ . ■

The bounds given here are rather simple and tight. If frequency splitting is not applied (the case when  $N = 1$ ), the mean local delay will surely be infinite. To guarantee a finite mean local delay, the value of  $N$  should be larger than or equal to two. It is valuable to investigate for which range of parameters the optimal value  $N_{\text{opt}}$  will be two. The following corollary gives the condition for  $N_{\text{opt}} = 2$ .

*Corollary 1:* If the intensity of transmitters  $\lambda$  satisfies the following inequality,

$$\lambda < \frac{\ln \frac{3}{2} - \frac{1}{6} \theta r_0^\alpha W N_0}{c_d r_0^d \theta^\delta C(\delta) (2^{-\delta} - 2^{\delta-1} 3^{-\delta})}, \quad (36)$$

the optimal number of sub-bands  $N_{\text{opt}}$  that minimizes the mean local delay will be 2.

*Proof:* Since we have proved that mean local delay  $D(N)$  is a function that first decreases and then increases with  $N$ , the condition for  $N_{\text{opt}} = 2$  is  $D(2) < D(3)$ . By substituting the expression of mean local delay (5) into  $D(2) < D(3)$ , we get the condition in the corollary. Notice that the right side of the inequality in (36) may be a negative, in which case the condition for  $\lambda$  cannot be satisfied. ■

In Fig. 5, we plot the optimal number of sub-bands  $N_{\text{opt}}$  and its bounds given in (30) as a function of the path loss exponent for different  $\theta$ . The optimal number  $N_{\text{opt}}$  is obtained by numerical calculation of the solution of the equation (33). We observe from Fig. 5 that the bounds are quite tight and give a excellent approximation of the value  $N_{\text{opt}}$ . This figure also shows that the optimal value  $N_{\text{opt}}$  decreases with increasing path loss exponent, which verifies our aforementioned discussion. The curves show that when the path loss exponent is fixed, the optimal number  $N_{\text{opt}}$  is an increasing function of the SINR threshold  $\theta$ , which can be also perceived from the expression (30). This is reasonable since with larger SINR threshold, the condition for successful transmission becomes harsher, and more sub-bands are needed to meet the stronger requirements. In Fig. 6, we plot the minimum value of the normalized mean local delay when the optimum number of sub-bands  $N_{\text{opt}}$  is used. We observe from Fig. 6 that there is an intersection between different curves, which means that the choice of SINR threshold  $\theta$  has direct impact on the mean local delay. In Section V, we will try to obtain the optimal SINR threshold.

## B. ALOHA

Since the expressions of the mean local delays for ALOHA and for FHMA are the same when  $p = 1/N$  and thermal noise ignored, we give the optimal transmit probability in the following theorem directly and omit the proof.

*Theorem 6:* The bounds of the optimal transmit probability which minimizes the mean local delay is

$$p_{\text{opt}} \in \left( \frac{1}{\lambda c_d r_0^d \theta^\delta C(\delta) + 2}, \frac{1}{\lambda c_d r_0^d \theta^\delta C(\delta)} \right). \quad (37)$$

## V. OPTIMAL SINR THRESHOLD $\theta$

In the discussion above, we have already derived tight bounds for the optimal number of sub-bands  $N_{\text{opt}}$  and the optimal transmit probability  $p_{\text{opt}}$  to

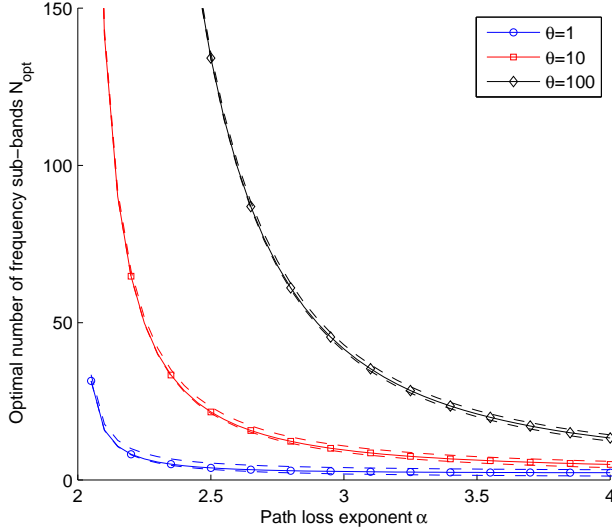


Fig. 5. Optimal number of sub-bands  $N_{\text{opt}}$  and its bounds  $(t_0, t_0+2)$  as a function of the path loss exponent  $\alpha$ .

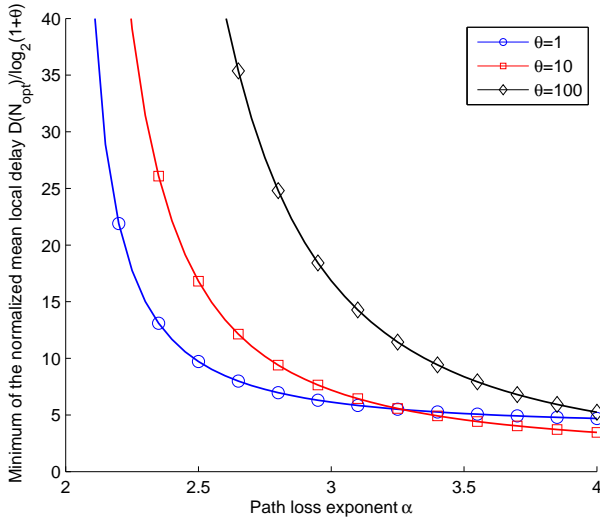


Fig. 6. Minimum of the normalized mean local delay  $\frac{D(N_{\text{opt}})}{\log_2(1+\theta)}$  as a function of the path loss exponent  $\alpha$ .

minimize the mean local delay when the SINR threshold  $\theta$  is fixed, and we also give accurate approximation to the actual optimal value. The following analysis will focus on deriving the optimal threshold  $\theta_{\text{opt}}$  or its bounds when the number of sub-bands  $N$  or the transmit probability  $p$  is fixed. However, as mentioned in Section II, the duration of each time slot is proportional to  $\frac{1}{\log_2(1+\theta)}$ . In order to characterize the actual delay, we slightly modify the optimization objective as the normalized mean local delay, i.e.,  $\frac{D(N)}{\log_2(1+\theta)}$  for FHMA and

$\frac{\tilde{D}(p)}{\log_2(1+\theta)}$  for ALOHA. In the following analysis, we consider two asymptotic regimes: the interference-limited regime and the noise-limited regime. The interference-limited regime is typically employed in cellular radio systems like CDMA networks, where the interference dominates over the thermal noise. The noise-limited regime is appropriate if the distance between concurrent transmitters is much larger than the distance of the typical link, in which case the interference in the network is negligible.

#### A. FHMA

The following theorem gives the optimal threshold  $\theta_{\text{opt}}$  or its bounds for FHMA in the interference-limited regime and noise-limited regime respectively.

*Theorem 7:* In the noise-limited regime, the optimal threshold  $\theta_{\text{opt}}$  that minimizes the normalized mean local delay  $\frac{D(N)}{\log_2(1+\theta)}$  for FHMA is given by

$$\theta_{\text{opt}} = \exp\left(\mathcal{W}\left(\frac{N}{r_0^\alpha W N_0}\right)\right) - 1, \quad (38)$$

where  $\mathcal{W}(z)$  is the Lambert  $\mathcal{W}$  function which solves  $\mathcal{W}(z)e^{\mathcal{W}(z)} = z$ . In the interference-limited regime, the bounds of  $\theta_{\text{opt}}$  are given by

$$\theta_{\text{opt}} \in \left(b_0^{-1/(\delta+1)} - 1, b_0^{-1/\delta}\right), \quad (39)$$

$$b_0 = \lambda c_d r_0^d \delta C(\delta) (N-1)^{\delta-1} N^{-\delta}.$$

*Proof:* The derivative of the normalized mean local delay with respect to  $\theta$  is

$$\frac{\partial \left(\frac{D(N)}{\log_2(1+\theta)}\right)}{\partial \theta} = \frac{h(\theta)\theta^{\delta-1}N}{\log_2(1+\theta)} \exp(\theta r_0^\alpha W N_0 N^{-1}) + \lambda c_d r_0^d \delta C(\delta) (N-1)^{\delta-1} N^{-\delta}, \quad (40)$$

where  $h(\theta)$  is as follows

$$h(\theta) = \lambda c_d r_0^d \delta C(\delta) (N-1)^{\delta-1} N^{-\delta} + r_0^\alpha W N_0 N^{-1} \theta^{1-\delta} - \frac{1}{\theta^{\delta-1}(1+\theta) \ln(1+\theta)}. \quad (41)$$

Next, we prove that  $h(\theta)$  is a strictly increasing function of  $\theta$ , then we show that the equation  $h(\theta) = 0$  has unique solution. Let  $l(\theta) = \theta^{\delta-1}(1+\theta) \ln(1+\theta)$  and the derivative of  $l(\theta)$  is as follows

$$\begin{aligned} l'(\theta) &= ((\delta-1)\theta^{\delta-2} + \delta\theta^{\delta-1}) \ln(1+\theta) + \theta^{\delta-1} \\ &> ((\delta-1)\theta^{\delta-2} + \delta\theta^{\delta-1})\theta + \theta^{\delta-1} \\ &> 0. \end{aligned}$$

Thus,  $l(\theta)$  is a strictly increasing function of  $\theta$ . This implies that  $h(\theta)$  is also a strictly increasing function of  $\theta$ . Since  $\lim_{\theta \rightarrow 0^+} h(\theta) = -\infty$  and  $\lim_{\theta \rightarrow \infty} h(\theta) = +\infty$ , the equation  $h(\theta) = 0$  has unique solution  $\theta_{\text{opt}}$  that minimizes the mean local delay.

In the noise-limited regime, the interference is ignored and the equation  $h(\theta) = 0$  has the form

$$r_0^\alpha W N_0 N^{-1} \theta^{1-\delta} - \frac{1}{\theta^{\delta-1}(1+\theta) \ln(1+\theta)} = 0. \quad (42)$$

Solving this equation we obtain (38).

In the interference-limited regime, the noise is ignored, and  $h(\theta) = 0$  has the form

$$\lambda c_d r_0^d \delta C(\delta) (N-1)^{\delta-1} N^{-\delta} - \frac{1}{\theta^{\delta-1}(1+\theta) \ln(1+\theta)} = 0. \quad (43)$$

A closed-form solution for the above equation does not exist. By applying the inequality  $\frac{\theta}{1+\theta} < \ln(1+\theta) < \theta$  to (43), we get the following inequality

$$\begin{aligned} \frac{1}{(1+\theta)^{\delta+1}} &< \frac{1}{\theta^\delta(1+\theta)} \\ &< \lambda c_d r_0^d \delta C(\delta) (N-1)^{\delta-1} N^{-\delta} < \frac{1}{\theta^\delta}. \end{aligned} \quad (44)$$

From the above inequality, we get the bounds in (39) ■

## B. ALOHA

Based on the similarity between the expressions of the normalized mean local delays for ALOHA and for FHMA, we obtain the optimal threshold for ALOHA directly.

**Theorem 8:** In noise-limited regime, the optimal threshold  $\theta_{\text{opt}}$  that minimizes the normalized mean local delay for ALOHA is as follows

$$\theta_{\text{opt}} = \mathcal{W} \left( \exp \left( \frac{1}{r_0^\alpha W N_0} \right) \right) - 1. \quad (45)$$

In interference-limited regime, the bounds of  $\theta_{\text{opt}}$  are given by

$$\begin{aligned} \theta_{\text{opt}} &\in \left( b_0^{-1/(\delta+1)} - 1, b_0^{-1/\delta} \right), \\ b_0 &= \lambda c_d r_0^d \delta C(\delta) p(1-p)^{\delta-1}. \end{aligned} \quad (46)$$

## VI. DESIGN INSIGHT

### A. Mean Delay-Jitter Tradeoff

In the system design, the jitter of delay, typically characterized by the variance, is a parameter as important as the mean local delay. The optimal value of  $N$  that minimizes the mean local delay is often not the one that minimizes the variance; thus there is a tradeoff between the mean and the variance of the local delay. Fig. 7(a) and Fig. 7(b) visualize the relationship between the mean and the variance of the normalized local delay for FHMA and for ALOHA respectively. From Fig. 7(a) we observe that in FHMA the favorable operating point has a reasonably wide tuning range because the variance stabilizes fast when  $N$  gets large. In contrast, we observe from Fig. 7(b) that in ALOHA the curves turn sharply.

### B. Tail Probability of the Local Delay

The tail probability, which characterizes the ccdf of the local delay, is an important measure of the system performance since one may require (as a QoS constraint) that the probability that the local delay exceeds a certain threshold is less than a predefined value. Based on the mean and variance of the local delay we have derived and by applying the one tailed Chebyshev's inequality, we obtain an upper bound for the tail probability.

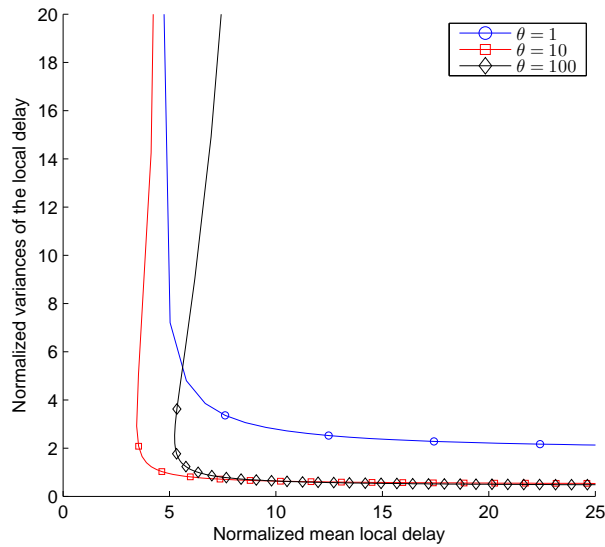
For example, in FHMA with  $N > 1$ , letting  $X = \sum_{i=1}^N \Delta_i$  be the local delay, the tail probability is upper bounded as follows

$$\mathbb{P}\{X > T_0\} \leq \frac{V(N)}{V(N) + (T_0 - D(N))^2}, (T_0 > D(N)). \quad (47)$$

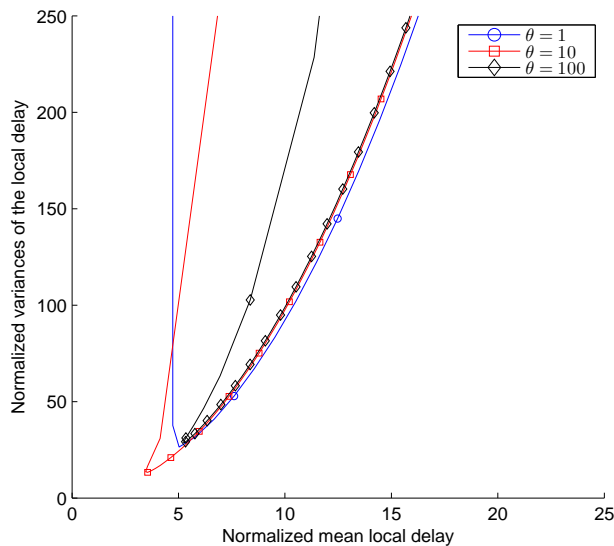
For example, if we let the design requirement be that the probability that the local delay exceeds 10 is less than 5%. Then, the threshold of the local delay is fixed as  $T_0 = 10$ , and the upper bound of the tail probability given by (47) with varying  $N$  is shown in Fig. 8. We observe that in order to achieve the probability 5%, the number of sub-bands  $N$  for the case when  $\theta = 1, 10, 100$  should be chosen larger than 15, 50, 95 respectively.

## VII. CONCLUSIONS

In this work, we studied the problem of reducing the effect of interference correlation by introducing different MAC dynamics. We derived the mean and



(a) FHMA



(b) ALOHA

Fig. 7. Mean delay-jitter tradeoff.

variance of the local delay and evaluated how the interference correlation can be reduced by FHMA and ALOHA. We also evaluated the optimal number of sub-bands in FHMA and the optimal transmit probability in ALOHA that minimize the mean local delay.

The results reveal that there exist two operation regimes for the network, i.e. correlation-limited regime and bandwidth-limited regime, which are separated by the optimal number of sub-bands in FHMA and the optimal transmit probability in ALOHA. If no MAC dynamics is employed, the

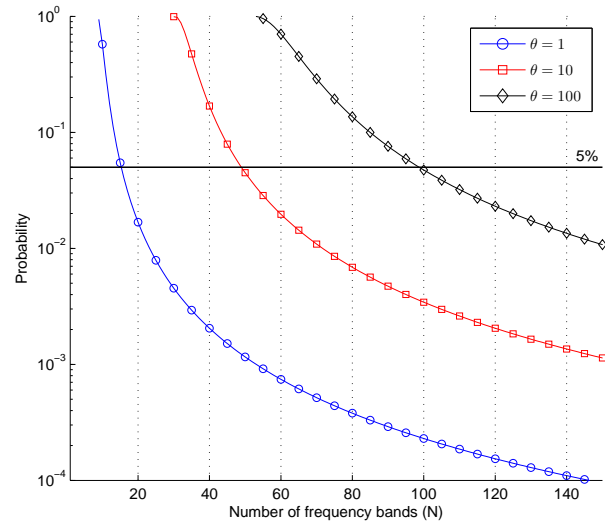


Fig. 8. Upper bounds of the tail probability of local delay given by (47) as a function of the number of sub-bands  $N$  when fixing  $T_0 = 10$  in FHMA.

local delay has a heavy tail distribution which results in infinite mean local delay; meanwhile, simply employing FHMA and ALOHA will greatly decrease the mean local delay. By comparing the results of FHMA and ALOHA, we observed that while the mean local delays of the two protocols are the same for certain parameters, the variances are rather different: when  $p = \frac{1}{N}$  and  $N \rightarrow \infty$ , the variance in FHMA tends to be a small constant, while in ALOHA the variance goes to infinity and scales as  $\Theta(N^2)$ . This observation shows that FHMA outperforms ALOHA if the difference in system costs is not taken into consideration; however, when considering the system costs, the overhead for FHMA is much greater than ALOHA because in FHMA each transmitter should inform the corresponding receiver that which sub-band to listen on.

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